

Glassy Behavior and Jamming of a Random Walk Process for Sequentially Satisfying a Constraint Satisfaction Formula

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Random K -satisfiability (K -SAT) is a model system for studying typical-case complexity of combinatorial optimization. Recent theoretical and simulation work revealed that the solution space of a random K -SAT formula has very rich structures, including the emergence of solution communities within single solution clusters. In this paper we investigate the influence of the solution space landscape to a simple stochastic local search process **SEQSAT**, which satisfies a K -SAT formula in a sequential manner. Before satisfying each newly added clause, **SEQSAT** walk randomly by single-spin flips in a solution cluster of the old subformula. This search process is efficient when the constraint density α of the satisfied subformula is less than certain value α_{cm} ; however it slows down considerably as $\alpha > \alpha_{cm}$ and finally reaches a jammed state at $\alpha \approx \alpha_j$. The glassy dynamical behavior of **SEQSAT** for $\alpha \geq \alpha_{cm}$ probably is due to the entropic trapping of various communities in the solution cluster of the satisfied subformula. For random 3-SAT, the jamming transition point α_j is larger than the solution space clustering transition point α_d , and its value can be predicted by a long-range frustration mean-field theory. For random K -SAT with $K \geq 4$, however, our simulation results indicate that $\alpha_j = \alpha_d$. The relevance of this work for understanding the dynamic properties of glassy systems is also discussed.

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I. INTRODUCTION

The random K -satisfiability (K -SAT) problem asks to determine whether or not a Boolean function with M clauses (each of which applying a constraint to K randomly chosen variables from a set of N Boolean variables) can be evaluated to be true. This problem has been studied extensively by computer scientists in the context of typical-case computation complexity of NP-complete combinatorial satisfaction and optimization [1]. Statistical physics has played an important role in understanding the energy landscape of the random K -SAT problem, especially the (zero-energy) solution space structure of a satisfiable random K -SAT formula [2, 3, 4, 5, 6]. It was revealed that [6], as the constraint density $\alpha \equiv M/N$ of a random K -SAT formula increases from zero, the solution space of the formula experiences a series of phase transitions before it finally becomes unsatisfiable as α exceeds a threshold value α_s . Two very important phase transitions in this series are the clustering or dynamic transition at $\alpha = \alpha_d$ (where the solution space splits into exponentially many dominating Gibbs states) and the condensation transition at $\alpha = \alpha_c \geq \alpha_d$ (where the number of dominating solution Gibbs states becomes sub-exponential). Statistical physics has also contributed a very efficient solver, survey-propagation (SP) [7, 8], for the random K -SAT problem. SP is able to find solutions for large random 3-SAT formulas with millions of variables even when the constraint density of the formulas approach closely the satisfiability threshold α_s . The magic power of SP is not yet fully understood. These and other theoretical and algorithmic achievements make the interface between theoretical computer science and statistical physics a prosperous research area [9, 10].

As its energy landscape and solution space structure both are very complex, the random K -SAT problem may serve as an interesting model system for studying glassy dynamics. For example, similar to lattice glass models [11] and spin-facilitated kinetic Ising models [12, 13], one may apply a random external field to each variable of a random K -SAT formula and investigate how the configuration of the variables changes with time under the hard constraint of the clauses (which are not allowed to violate during the configuration evolution process). For a single-spin-flip dynamical process, the configuration of the system is confined to a connected component (a single solution cluster [14]) of the solution space of the K -SAT formula. However, the dynamics within such a solution cluster is not necessarily trivial. Simulations performed on random 3-SAT and 4-SAT formulas revealed that, as the constraint density α becomes relatively large, the structure of a single solution cluster of the formula is no longer homogeneous [15]. The solutions may aggregate into many communities, each of which containing a group of solutions that are densely connected to each other while the connections between two different communities are relatively sparse. In a community-rich solution cluster, a local search process will get trapped into a solution community for some time before it passes through a bridge region and enters into another different community. Such an entropic trapping effect will result in multiple time scales in the relaxation dynamics, which are typical of glassy systems.

In this work we study the dynamics of a simple process **SEQSAT** which performs an unbiased random walk in a

solution cluster of a random K -SAT subformula F_m . The subformula F_m contains the first m clauses ($m = 0$ initially) of a long random K -SAT formula F of N variables. As soon as **SEQSAT** reaches a solution of F_m that also satisfies the $(m + 1)$ -th clause of formula F , this clause is added to subformula F_m and **SEQSAT** starts walking randomly in the solution cluster of the enlarged subformula F_{m+1} again. The waiting time needed for **SEQSAT** to find a solution that satisfies the $(m + 1)$ -th clause gives a measure of the viscosity in moving in the solution cluster of the old subformula of m clauses. We find that, when the constraint density $\alpha = m/N$ of the satisfied subformula F_m is small, the waiting time to satisfy the next clause is usually small. However the average value of the waiting time increases considerably as α exceeds a threshold value α_{cm} ; and finally when α reaches a larger threshold value α_j , the average waiting time essentially diverges and **SEQSAT** stops to satisfy the next clause. The dramatic slowing down of **SEQSAT** at $\alpha \geq \alpha_{cm}$ is understood in terms of the complex community structures in the explored single solution clusters of the subformula F_m . We also calculate the mean value of the jamming transition point α_j by a long-range frustration mean-field theory [16, 17], and find that the theoretical prediction is in good agreement with simulation results in the case of random 3-SAT. For $K = 3$, α_j is larger than the solution space clustering transition point α_d , while for $K \geq 4$ it appears that α_j coincides with α_d .

Recently the idea of constructing solutions for a constraint satisfaction formula by adding constraints one after another was explored by Krzakala and Kurchan [18]. The present work differs from Ref. [18] in that the **SEQSAT** process never allows the energy of the system to increase, while the **WALKCOL** algorithm used in Ref. [18] for the random Q -coloring problem is able to cross energy barriers. The **SEQSAT** process is also different from the **CHAINSAT** process of Alava and co-authors [19]. Although **CHAINSAT** also prohibits any energy increases, it may satisfy a clause m at the price of unsatisfying a clause m' that was previously satisfied. In the **SEQSAT** process, however, a satisfied clause will remain to be satisfied. The performance of **SEQSAT** may be further improved if such a hard constraint can be made more softer by introducing a positive temperature parameter. In this paper we also study the performance of a biased random walk search process.

II. THE RANDOM K -SATISFIABILITY PROBLEM

A K -SAT formula F contains N variables ($i = 1, 2, \dots, N$) and M clauses ($a = 1, 2, \dots, M$). Each variable i has a binary state $\sigma_i = \pm 1$, and each clause a represents a constraint which involves a subset $\partial a = \{i_1, i_2, \dots, i_K\}$ of the N variables whose size $|\partial a| \equiv K$. The energy of a spin configuration $\vec{\sigma} \equiv \{\sigma_1, \sigma_2, \dots, \sigma_N\}$ is defined as

$$E(\vec{\sigma}) = \sum_{a=1}^M E_a, \quad (1)$$

with the energy of clause a being

$$E_a = \prod_{i \in \partial a} \frac{1 - J_a^i \sigma_i}{2}. \quad (2)$$

In Eq. (2), J_a^i is the recommended spin value to variable $i \in \partial a$ by clause a . If at least one of the variables $i \in \partial a$ takes the recommended value $\sigma_i = J_a^i$, then $E_a = 0$ and clause a is said to be satisfied. Clause a is violated and $E_a = 1$ if no variables $i \in \partial a$ takes the recommended value J_a^i .

In constructing a random K -SAT formula, the K variables in the set ∂a of each clause a are randomly chosen from the N variables, and for each variable $i \in \partial a$ the preferred spin value J_a^i is set to be $+1$ or -1 with equal probability. Given a random K -SAT formula one needs to find, among the total number of 2^N configurations, at least one configuration $\vec{\sigma}$ that satisfies all the clauses ($E(\vec{\sigma}) = 0$), or to prove that no such satisfying configurations exist. The random K -SAT problem is an extensively studied model in the context of typical-case computation complexity. When the number of variables N is very large, a random K -SAT formula has a high probability to be satisfiable (respectively, unsatisfiable) if the constraint density $\alpha = M/N$ is less (respectively, greater) than certain threshold value $\alpha_s(K)$. For $K = 3$, Kirkpatrick and Selman estimated $\alpha_s \simeq 4.2$ through extensive numerical simulations [1], while the mean-field theory of statistical physics predicts that $\alpha_s(3) = 4.2667$ [4, 20]. The theory of Mezard and co-authors [4] is also applicable to the random K -SAT problem with $K \geq 4$.

There are many algorithms for finding solutions for a random K -SAT formula. A widely used one is the branch and bound DPLL algorithm of Davis, Putnam, Logemann, and Loveland (one can refer to the review article [21] for the major developments on the DPLL algorithm). Frieze and Suen [22] (see also [23] and the review paper [24]) showed that for random 3-SAT with constraint density $\alpha < 3.003$, the generalized unit clause heuristic has a high probability of constructing a solution in a single descent of the DPLL tree. But for $\alpha > 3.003$, backtracking is needed and this may result in an exponential increase of computation time [23]. In the case of $\alpha > 3$, many stochastic search

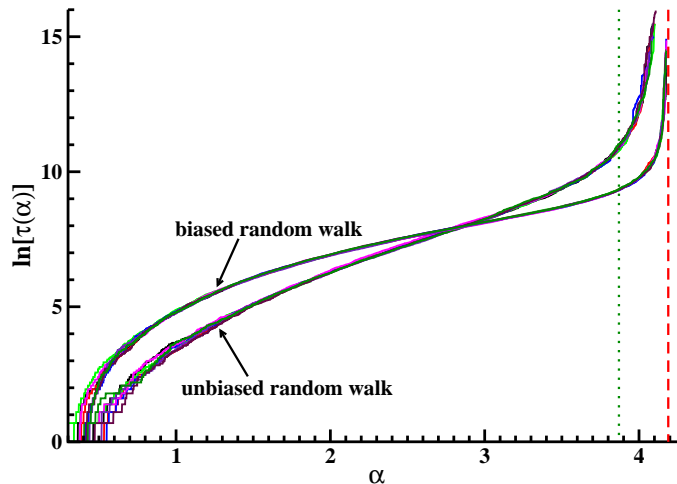


FIG. 1: Logarithm of the search time $\tau(\alpha)$ by SEQSAT to satisfy the first αN clauses of a random 3-SAT formula of $N = 10^5$ variables. The solid lines are results obtained from 16 simulation trajectories from different initial spin configurations (half of the trajectories are generated by an unbiased random walk rule, while the remaining half are by a biased random walk rule). The red dashed line corresponds to the jamming transition point $\alpha_j^\infty = 4.189724$ as predicted by the long-range frustration mean-field theory [17], the green dotted line marks the cluster transition point α_d [6], and the right-most black solid line marks the satisfiability threshold α_s [7, 8].

algorithms are able to outperform the DPLL algorithm in average computation time, examples are WALKSAT [25] (an enhanced version of RANDOMWALKSAT [26]), ASAT [27], CHAINSAT [19], belief-propagation [28, 29] and its variants (such as reinforcement [30]), and SP [7, 8].

Mean-field theory of statistical physics [6] predicted that the solution space of a large random K -SAT formula experiences several structural transitions as the constraint density α increases. At the clustering transition point $\alpha = \alpha_d$, the solution space of the formula splits into an exponential number of Gibbs states of equal importance. Later at the condensation transition point $\alpha = \alpha_c$ the solution space becomes dominated by only a few Gibbs states (for the special case of $K = 3$, the clustering and the condensation transition coincide). It was also found that some variables will become frozen to the same spin value in all the solutions of a Gibbs state [31, 32], this fact has serious consequences for stochastic local search algorithms.

III. RANDOM WALK PROCESSES

Several slightly different random walk processes were exploited in Refs. [15, 33] to study the solution space structure of single random K -SAT formulas. Here, based on the same idea of random walking, we present a simple stochastic solver SEQSAT for the random K -SAT problem. Given a random K -SAT formula F with N variables and M clauses with index $a = 1, 2, \dots, M$, we denote by F_m the subformula which contains the N variables and the first m clauses. The constraint density of this subformula is $\alpha = m/N$. We set $m = 0$ at time $\tau = 0$ and satisfy this clause-free subformula F_0 by randomly picking a configuration $\vec{\sigma}(0)$ from the whole set of 2^N configurations. Suppose the m -th clause is first satisfied at time $\tau(m)$, and the configuration at this time is $\vec{\sigma}(\tau(m))$. We then add the $(m+1)$ -th clause, and if this clause is satisfied by $\vec{\sigma}(\tau(m))$, we set $\tau(m+1) = \tau(m)$ and $\vec{\sigma}(\tau(m+1)) = \vec{\sigma}(\tau(m))$; otherwise we perform an *unbiased* random walk of single-spin flips starting from $\vec{\sigma}(\tau(m))$ in the solution space of subformula F_m [34] until a configuration $\vec{\sigma}(\tau(m+1))$ that also satisfies the $(m+1)$ -th clause is first reached at time $\tau(m+1) = \tau(m) + \mathcal{N}_{m+1}$, with $\mathcal{N}_{m+1} \times N$ being the total number of spin flips used to reach $\vec{\sigma}(\tau(m+1))$ from the solution $\vec{\sigma}(\tau(m))$. The waiting time of satisfying the $(m+1)$ -th clause is identified to be \mathcal{N}_{m+1} .

Starting from a random initial configuration $\vec{\sigma}(0)$, SEQSAT satisfies the clauses of a K -SAT formula F in a sequential order without overcoming any energy barriers. Simulation results of the next section demonstrate that such a simple steepest-descent algorithm actually has very good performance for single random K -SAT formulas. There are various possibilities for further improving SEQSAT. One extension is to introduce a small positive temperature T into the search process. A proposed spin flip is accepted with probability $\exp(-\Delta E/T)$, where ΔE is the change in the configuration energy due to this spin flip. At finite temperatures, the random walker can overcome not only solution space entropic barriers but also energetic barriers, therefore its dynamics will be even richer. Another extension, which we explore

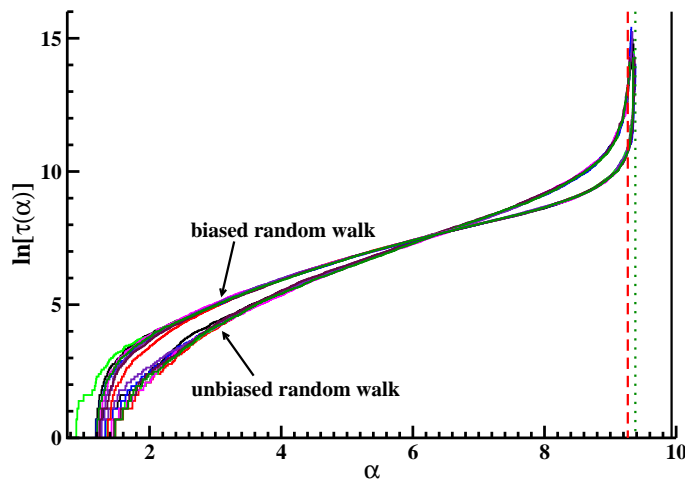


FIG. 2: Same as Fig. 1, but for a random 4-SAT formula of $N = 10^5$ variables.

in this paper, is to use biased random walk when traveling from one solution to a nearest-neighboring one.

The following biased random walk jumping scheme is used. Suppose in a configuration $\vec{\sigma}(t)$ at time t there are n flippable variables. We divide these n variables into two sets A and B : A contains all the variables that are not yet flipped since being flippable for the last time, and B contains all the flippable and flipped variables. If set A is non-empty, a variable is uniformly randomly chosen from A and its spin value is flipped, otherwise, a variable in set B is uniformly randomly chosen and flipped. Once a variable becomes flippable (because all its nearest-neighbor clauses are being satisfied by other variables), it has a larger probability to be flipped under the biased random walk rule as compared with the previously mentioned unbiased rule. This gives the random walker a preference to explore new regions of the solution space. As we show in the next section, however, this change of local search rule does not bring in qualitative improvement in search performance.

IV. SIMULATION RESULTS AND INTERPRETATION

A. The case of $K = 3$

The time $\tau(\alpha)$ needed by the unbiased SEQSAT random walk algorithm to satisfy sequentially the first αN clauses of a random K -SAT formula with $N = 10^5$ variables is shown in Fig. 1 for $K = 3$. When the constraint density α of the satisfied subformula is very low, SEQSAT satisfy every newly added clause almost immediately; as more clauses are added into the satisfied subformula, SEQSAT has to make some local adjustments of the spin configuration to satisfy a new clause, which takes some time, but the dynamics is still very efficient. However as the constraint density of the subformula is beyond $\alpha \approx 3.8$, the unbiased SEQSAT slows down considerably and then essentially stops to satisfy the newly added clause as α becomes even larger. For example, at $\alpha \approx 4.1$ it can take several weeks for the unbiased SEQSAT to satisfy a new clause. As a comparison, the satisfiability threshold of the random 3-SAT problem is $\alpha_s \approx 4.27$ [7, 20].

We have observed the same dynamic behavior as shown in Fig. 1 when performing the same unbiased SEQSAT simulation on different random 3-SAT formulas. The search time $\tau(\alpha)$ of satisfying the first αN clauses is found to scale linearly with the number N of variables when $N \geq 10^3$. Then for random 3-SAT formulas with different values of N , the curves of $\ln(\tau/N)$ as a function of α can be superimposed onto each other. The diverging behavior of $\tau(\alpha)$ for large values of constraint density α suggests that walking within a solution cluster of the subformula F_m becomes more and more viscous as α increases. According to our opinion, the main reason of this viscosity increase is the emergence of complex community structure in the solution cluster of the subformula F_m as revealed by Ref. [15] (see Fig. 5). To satisfy the newly added $(m+1)$ -th clause, at least one of the attached K variables of the clause should be flipped. However, in order to make the necessary configuration rearrangement so that one of these K variables can finally be flipped, SEQSAT may have to travel through many local communities of solutions, each of them trapping the random walk process for a period of time. Such multiple trappings will make flipping a given variable a very difficult task.

The whole solution space of a K -SAT formula can be represented as a graph in which each vertex denotes a solution and each edge between a pair of vertices means the two corresponding solutions are related by a single-spin flip [14]. A

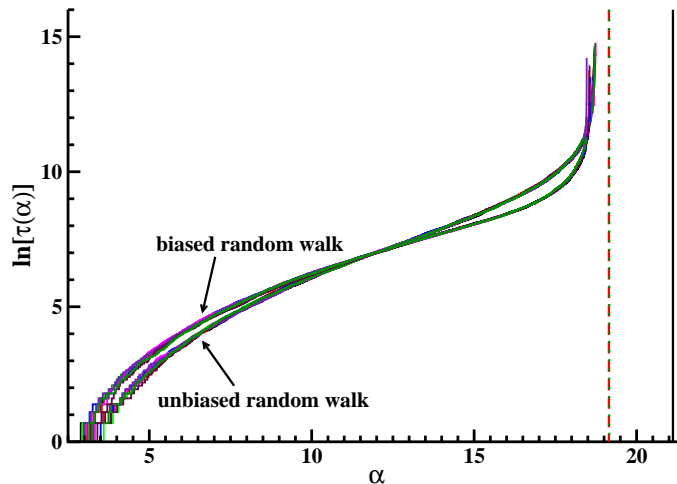


FIG. 3: Same as Fig. 1, but for a random 5-SAT formula of $N = 10^5$ variables.

connected component (a solution cluster) of such a graph contains all the solutions that are reachable from a reference solution by a sequence of single-spin flips. In the **SEQSAT** process, after a solution $\vec{\sigma}$ that satisfies the first m clauses of the random K -SAT formula F is reached, the $(m+1)$ -th clause is added and **SEQSAT** performs a random walk of single spin flips starting from $\vec{\sigma}$ in the solution cluster of subformula F_m until a configuration $\vec{\sigma}'$ which also satisfies the newly added clause is reached (i.e., the subformula F_{m+1} is now satisfied). Both $\vec{\sigma}$ and $\vec{\sigma}'$ belong to the same solution cluster (say C_m) of subformula F_m . As **SEQSAT** always tries to satisfy a newly added clause within the solution cluster of the satisfied old subformula, the solution cluster C_{m+1} of subformula F_{m+1} reached by **SEQSAT** is a subset of C_m , i.e.,

$$C_0 \supseteq C_1 \supseteq \dots \supseteq C_m \supseteq C_{m+1} \supseteq \dots \quad (3)$$

According to the numerical studies of Refs. [15, 33], the connection pattern of a single solution cluster C_m of a random K -SAT subformula F_m may be quite heterogeneous. Some of the solutions may be densely inter-connected with each other but are only very sparsely connected to the other solutions of the same cluster. These solutions then form a solution community (Fig. 5). Different communities of the same solution cluster are linked together by inter-community edges and/or single solutions that lie at the borders of several different communities. In a community-rich solution cluster C_m of subformula F_m , some of the communities (e.g., community A in Fig. 5) may only contain solutions that do not satisfy the $(m+1)$ -th clause of formula F . If **SEQSAT** starts unfortunately from a solution of such a community A , it then has to waste some time wondering along the internal edges of community A until it finally jumps onto an edge that leads to another different community B . (Even if the community A contains some solutions that satisfy the $(m+1)$ -th clause, if the fraction of such solutions in community A is very small, **SEQSAT** still will take time to reach them.)

If the above-mentioned entropy trapping effect of solution communities is the major reason for the slowing down of **SEQSAT**, then the constraint density value α at which $\tau(\alpha)$ begins to increase rapidly should be close to the value of α at which community structure begins to appear in a single solution cluster of a random K -SAT formula. When running **SEQSAT** on random K -SAT formulas, we can define an empirical threshold value α_{cm} as the *first* constraint density at which **SEQSAT** takes \mathcal{N}_{cm} time units (i.e., $\mathcal{N}_{cm} \times N$ single spin flips) or more to satisfy the next clause. Of course α_{cm} is a random variable that takes slightly different values in different trajectories of **SEQSAT** on the same formula. For the eight unbiased random walk trajectories shown in Fig. 1, we find that $\alpha_{cm} \approx 3.37$ if we set $\mathcal{N}_{cm} = 100$ and $\alpha_{cm} \approx 3.82$ if we set $\mathcal{N}_{cm} = 1000$. These results are consistent with the prediction of Ref. [35] that the solution space of the random 3-SAT problem is heterogeneous and community-rich at constraint density $\alpha \geq 3.75$. The heterogeneity of the solution space of the random 3-SAT problem causes the slowing down of the unbiased **SEQSAT** local search process.

A clustering transition occurs in the solution space of the random 3-SAT problem at $\alpha = \alpha_d = 3.87$, with the solution space breaks into exponentially many solution clusters [6]. As shown in Fig. 1, when $\alpha > \alpha_d$ the unbiased **SEQSAT** process is still able to find solutions for a random 3-SAT formula. The solution space ergodicity-breaking transition therefore does not lead to divergence of the search time of **SEQSAT**. Similar phenomena were observed in previous studies [18, 19].

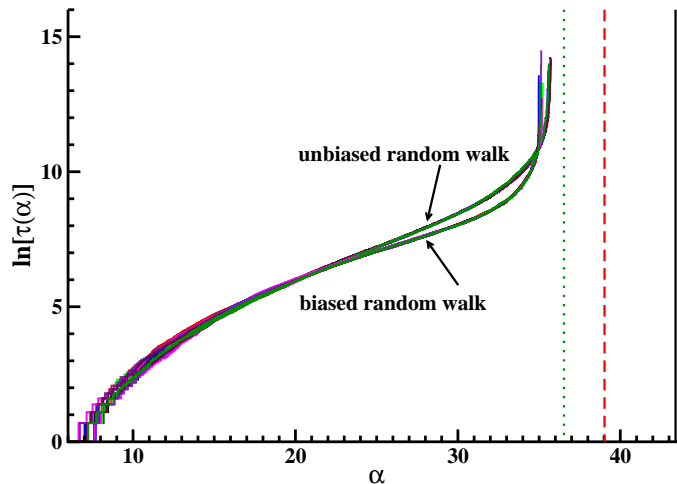


FIG. 4: Same as Fig. 1, but for a random 6-SAT of $N = 10^5$ variables.

B. The cases of $K \geq 4$

The simulation results of the unbiased SEQSAT process for random K -SAT formulas are shown in Fig. 2 ($K = 4$), Fig. 3 ($K = 5$) and Fig. 4 ($K = 6$). Similar to the results obtained for the random 3-SAT formula, as the constraint density α becomes large the SEQSAT process slows down exceedingly in the cases of $K \geq 4$. Compared with Fig. 1 of the $K = 3$ case, the main difference for $K \geq 4$ is that the search time of the unbiased SEQSAT appears to diverge at the solution space clustering transition point $\alpha = \alpha_d$.

This difference may be understood by recalling that at and after the clustering transition point α_d , the solution space of a random 3-SAT formula is still dominated by a few largest clusters, while that of a random K -SAT ($K \geq 4$) formula is dominated by an exponential number of relatively small clusters [6]. The spin values of a finite fraction of the variables may be frozen in each of these small clusters of a random K -SAT formula ($K \geq 4$), and consequently a newly added clause has a large probability to be unsatisfied by all the solutions in the solution cluster.

C. Comparing biased and unbiased random walk search processes

Figures 1-4 also compare the performance of the biased random walk search process with that of the unbiased random walk search process. For random 3- and 4-SAT formulas, the biased SEQSAT process is more efficient than the unbiased process, but both processes appear to diverge at the same critical constraint density values. For random 5- and 6-SAT formulas, on the other hand, the divergence point of the biased SEQSAT process is smaller than that of the unbiased process.

V. LONG-RANGE FRUSTRATION THEORY ON THE JAMMING TRANSITION

The simulation results of the preceding section demonstrate that, the random walk searching process SEQSAT, being prohibited from jumping between different solution clusters of a K -SAT subformula F_m , will eventually reach a jammed state after the first $m_{max} = \alpha_j N$ clauses of the original random formula F have been satisfied. SEQSAT is unable to satisfy the $(m_{max} + 1)$ -th clause if in the reached solution cluster of $F_{m_{max}}$, the variables that are involved in this clause are all frozen to the ‘wrong’ spin value. The critical constraint density $\alpha_j = m_{max}/N$ (the jamming transition point) may be different in different runs of SEQSAT, as different runs of SEQSAT may reach different solution clusters of the solution space. Our simulation results reveal that the jamming constraint densities α_j as obtained for many trajectories of SEQSAT on the same random K -SAT formula and on different random K -SAT formulas are very close to each other. It is anticipated that, in the thermodynamic limit of $N \rightarrow \infty$, the SEQSAT process has a true jamming transition at a critical constraint density α_j^∞ .

The jamming transition of SEQSAT is closely related to the freezing of variables in a solution cluster of a random K -SAT formula. Recently there were several interesting studies on the freezing transition of the solution space of the random K -SAT problem [31, 32, 36, 37]. For example, the freezing transition point for the random 3-SAT problem is

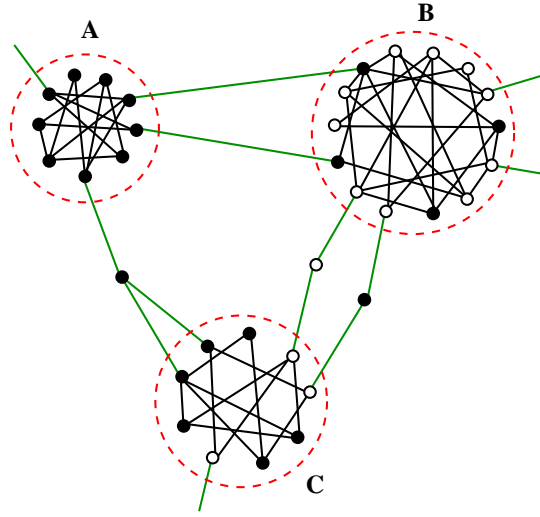


FIG. 5: Schematic diagram for the connection pattern of a single solution cluster C_m of a random K -SAT subformula F_m (containing the first m clauses of a larger formula F). Circles represent solutions, and edges represent a single-spin flips between two solutions of unit Hamming distance [15]. Those solutions that do not satisfy the $(m+1)$ -th clause of formula F are denoted by filled circles, while the remaining solutions of C_m are all denoted by empty circles. Solutions in cluster C_m are grouped into many communities (A , B , C , etc.) such that a solution of a community is connected to many other solutions of the same community but are not or only very sparsely connected to solutions of other communities.

located at $\alpha = \alpha_f = 4.254$ [32], while that for the random 4-SAT problem is located at $\alpha_f = 9.88$ [31]. These threshold constraint densities correspond to the appearance of frozen variables in the dominating Gibbs states of the solution space. On the other hand, the final solution clusters reached by the SEQSAT process should not be the dominating clusters of the solution space (otherwise, the $(m+1)$ -th clause would be satisfiable). Therefore it is natural to expect (and is confirmed by simulation results shown in Figs. 1 and 2) that the jamming transition point α_j^∞ of SEQSAT is less than the freezing transition point α_f .

As shown in Figs. 2-4, the jamming transition point α_j for random K -SAT formulas with $K \geq 4$ are very close to the critical value α_d . On the other hand, for random 3-SAT formulas, α_j is much larger than α_d . We now try to predict the value of α_j^∞ as a function of K using the long-range frustration theory of Refs. [16, 17]. In a solution cluster of a random K -SAT formula with $N \rightarrow \infty$ variables and constraint density α , some of the variables are unfrozen as their spin values are positive in some of the solutions of this cluster and negative in the remaining variables. We denote the fraction of unfrozen variables in this solution cluster as q_0 . With respect to a pre-specified spin value σ_i^* , a unfrozen variable i can be regarded as either type-I unfrozen or type-II unfrozen [17]. If flipping variable i to $\sigma_i = \sigma_i^*$ leads to the fixation of the spin values of a finite fraction of all the other unfrozen variables, then variable i is type-I unfrozen; if setting $\sigma_i = \sigma_i^*$ only affects a small number of other unfrozen variables, then i is type-II unfrozen. We denote by R the probability that a randomly chosen unfrozen variable i is type-I unfrozen with respect to a randomly specified spin value σ_i^* .

The spin states of the type-I unfrozen variables of the random K -SAT formula are strongly correlated. If one fix the spin of one such variable, the final effect might be that the spins of a large fraction of all the other type-I unfrozen variables are also being fixed. A type-I unfrozen variable takes different spin values in different dominating communities, but within each community its spin is frozen to one value.

As one adds more clauses to the random K -SAT formula, the solution cluster shrinks, and then both q_0 and R will change. According to Ref. [17] the following set of self-consistent equations can be derived:

$$q_0 = \sum_{m=0}^{\infty} (P_v(m))^2, \quad (4)$$

$$R = 1 - \exp(-\lambda_3 R). \quad (5)$$

In the above two equations, the function $P_v(m)$ is defined by

$$P_v(m) = \sum_{n=0}^m f(n, \lambda_2) P_f(m-n) \quad (6)$$

with

$$P_f(n) = f(2n, \lambda_1) C_{2n}^n 2^{-2n} + \sum_{s=2n+1}^{\infty} f(s, \lambda_1) C_s^n 2^{1-s},$$

$$f(n, \lambda) = e^{-\lambda} \lambda^n / n!,$$

and the three λ parameters are expressed as

$$\begin{aligned} \lambda_1 &= K\alpha((q_0 R + 1 - q_0)/2)^{K-1} - ((1 - q_0)/2)^{K-1}, \\ \lambda_2 &= (K\alpha/2)((1 - q_0)/2)^{K-1}, \\ \lambda_3 &= (K(K-1)\alpha q_0(1-R)/2)((1 - q_0)/2)^{K-2}. \end{aligned}$$

K	α_j^∞	α_d [6]	α_s [20]	q_0^j
3	4.1897	3.87	4.2667	0.5270
4	9.2653	9.38	9.931	0.3994
5	19.1480	19.16	21.117	0.3359
6	39.0269	36.53	43.37	0.2967
7	79.4245		87.79	0.2694
8	161.78			0.2479
9	329.704			0.230
10	671.796			0.2147
11	1368.01			0.2015
12	2783.8			0.190

TABLE I: α_j^∞ is the threshold value of jamming transition as predicted by the long-range frustration theory [17], α_d is the clustering transition point reported by Ref. [6], α_s is the satisfiability transition point [20], and q_0^j is the fraction of unfrozen variables at the jamming transition as predicted by the long-frustration frustration theory.

The jamming transition point α_j^∞ corresponds to the smallest value of α at which a fixed point $q_0 < 1$ of Eqs. (4) and (5) first appears. Table I lists the value of α_j^∞ for $3 \leq K \leq 12$ and the corresponding fraction of unfrozen variables q_0^j at the jamming transition. As a comparison with simulation results, we have denoted by a red dashed line the predicted jamming transition point α_j^∞ in Figs. 1-4. For $K = 3$, we find $\alpha_j^\infty = 4.19$, which is larger than the clustering transition point $\alpha_d = 3.87$ but is in agreement with the simulation results of Fig. 1; while for $K = 4$, $\alpha_j^\infty = 9.27$ is smaller than the dynamic transition $\alpha_d = 9.38$. As Fig. 2 shows, for random 4-SAT formulas the **SEQSAT** process is able to reach constraint density values higher than α_j^∞ . The long-range frustration mean-field theory appears to give a satisfactory prediction of the jamming transition point of **SEQSAT** for random 3-SAT formulas but fails in the case of $K = 4$ and $K = 6$.

The solution space structure of the random 3-SAT problem is qualitatively different from that of the random K -SAT problem with $K \geq 4$ [6]. Beyond the clustering transition point α_d , the solution space of a large random 3-SAT formula is dominated by only a sub-exponential number of solution clusters, while that of a large random K -SAT ($K \geq 4$) is divided into an exponential number of solution clusters of equal statistical importance. The full-step replica-symmetry-breaking mean-field theory is needed to fully describe the statistical property of the solution space of the random 3-SAT problem $\alpha > \alpha_d(3)$ [38], but for the random K -SAT problems with $K \geq 4$, a simpler first-step replica-symmetry-breaking theory is believed to be sufficient. As the solution space of a random K -SAT ($K \geq 4$) formula has exponentially many communities or clusters at the vicinity of the clustering transition α_d , some of the assumptions of the long-range frustration mean-field theory may no longer be appropriate.

Figure 6 shows how the predicted jamming transition point α_j^∞ scales with K . The data is consistent with

$$\alpha_j^\infty(K) = 2^K \ln 2 + O(1). \quad (7)$$

Notice that the satisfiability threshold $\alpha_s(K)$ also the same scaling behavior [20]. The jamming value $\alpha_j^\infty(K)$ may serve as a good lower bound for the satisfiability threshold of the random K -SAT problem.

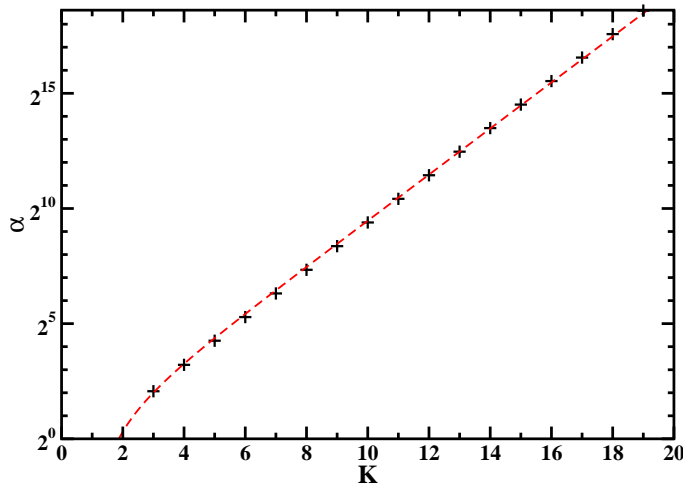


FIG. 6: The scaling behavior of the predicted jamming transition point α_j^∞ with K . The red dashed line is a fitting curve of the form $\alpha_j^\infty(K) = 2^K \ln 2 - c_0$, with the fitting parameter being $c_0 = -1.5 \pm 0.2$.

VI. CONCLUSION AND FURTHER DISCUSSIONS

In this work the dynamic behavior of a simple stochastic search algorithm **SEQSAT** for the random K -SAT problem were investigated by computer simulations. This simple algorithm is able to find solutions for a large random K -SAT ($K \geq 3$) formula if the constraint density α is less than certain threshold $\alpha_j(K)$, but it experiences a jamming transition as α approaches $\alpha_j(K)$ from below. For $K \geq 4$, we found that the jamming point $\alpha_j(K)$ is very close to the solution space clustering transition point $\alpha_d(K)$, but the jamming point $\alpha_j(3) \approx 4.19$ for the special case of $K = 3$ exceeds $\alpha_d(3) = 3.87$ considerably. We argued in this work that, the dramatic slowing down of **SEQSAT** at α close to the jamming point $\alpha_j(K)$ is caused by the entropic trapping effect of various solution communities in a single solution cluster of the random K -SAT formula. We also estimated the jamming transition point $\alpha_j(K)$ using the mean-field long-range frustration theory of Refs. [16, 17], and found that the calculated value of $\alpha_j(3)$ is in good agreement with simulation results.

The rapid increase of the search time $\tau(\alpha)$ of the dynamic process **SEQSAT** at α close to the jamming point α_j is reminiscent of the rapid increase of viscosity of a glass-forming liquid at low temperatures. These glassy behaviors may be governed to a large extent by the same physical mechanisms. The random K -SAT problem might serve as a very rich model system to study the connection between complex energy landscapes and glassy dynamics. In the present paper, the dynamics of **SEQSAT** is confined to a single connected component of the zero-energy ground-state configuration space of a K -SAT formula; in future studies, one may introduce external fields and/or a finite temperature to the system to observe more complex dynamic behaviors.

When the constraint density α of the random K -SAT formula is slightly beyond the jamming point α_j , the formula still contains exponentially many solutions, but **SEQSAT** is unable to reach any one of them. On the algorithmic side, a major limitation of **SEQSAT** is that it only explores a single connected component of the solution space. One may incorporate the energy-barrier crossing techniques of other heuristic algorithms into **SEQSAT** to enhance its performance. We also demonstrated that the efficiency of searching within a solution cluster can be elevated to some extent by using biased random walks [15, 33], but such a small change of local search rule does not lead to a shift of the jamming point $\alpha_j(K)$ to larger values.

Although the mean-field long-range frustration theory [17] is able to give good predictions on the jamming transition point $\alpha_j(K)$ for $K = 3$, it fails to do so for $K = 4, 6$. For $K \geq 4$ the jamming transition point of **SEQSAT** probably is identical to the clustering transition point α_d . There is an important uncontrolled approximation in the mean-field theory, namely that two type-I unfrozen variables have probability *one-half* of being prohibited from taking simultaneously their canalizing spin values [16, 17]. This approximation may not be very appropriate for the case of $K > 3$. On the hand, as shown in Table I, the predicted jamming transition point $\alpha_j(K)$, whose value is close to $\alpha_d(K)$, is always lower than the satisfiability threshold $\alpha_s(K)$ and has the same scaling behavior of $\alpha_j(K) \approx 2^K \ln(2)$ as $\alpha_s(K)$.

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